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DISCONTINUOUS "VISCOSITY" SOLUTIONS OF A DEGENERATE PARABOLIC EQUATION

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ABSTRACT. We study a nonlinear degenerate parabolic equation of the second order. Regularizing the equation by adding some artificial viscosity, we construct a generalized solution. We show that this solution is not necessarily continuous at all points.

1. INTRODUCTION

We consider the problem

$$\text{(I)} \qquad \begin{cases} u_t = u\Delta u - \gamma|\nabla u|^2 & \text{in } Q = \mathbf{R}^N \times \mathbf{R}^+, \qquad (1.1) \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where γ is a nonnegative constant and the initial function u_0 satisfies

$$\text{(H1)} \qquad u_0 \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \quad \text{and} \quad u_0 \geq 0 \quad \text{in } \mathbf{R}^N.$$

Equation (1.1) arises in several applications in biology and physics. References can be found in [2, 5, 8]. Equation (1.1) is of degenerate parabolic type: it loses its parabolicity at points where $u = 0$. Therefore Problem I does not in general have classical solutions, and we define solutions in a generalized sense.

Definition 1.1. $u \in L^\infty(Q) \cap L^2_{\mathrm{loc}}([0, +\infty); H^1_{\mathrm{loc}}(\mathbf{R}^N))$ is called a solution of Problem I if $u \geq 0$ almost everywhere in Q and

$$\int_{\mathbf{R}^N} u_0\psi(0)\,dx + \iint_Q (u\psi_t - u\nabla u \cdot \nabla\psi - (1+\gamma)|\nabla u|^2\psi)\,dx\,dt = 0$$

for any $\psi \in C^{1,1}(\overline{Q})$ with compact support in $\overline{Q}$.

In §2 we shall construct a solution of Problem I. This construction is based on the well-known viscosity method: we add a term $\varepsilon\Delta u$ $(\varepsilon > 0)$ to the right-hand side of equation (1.1) and let $\varepsilon \searrow 0$. It turns out that this limiting procedure defines exactly one limit function $u(x, t)$. In §6 we shall show that $u(x, t)$ is a solution of Problem I and we call $u(x, t)$ the "viscosity solution" of Problem I. However, there might be other solutions of Problem I. In fact, it has been shown that solutions of Problem I are not uniquely determined by the initial

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function u_0 (see [1, 5, 8]), but in this paper we focus our attention on the viscosity solution and in particular its regularity properties. Our "definition" of the viscosity solution is, although quite natural, rather ad hoc. It is based on the very special property of equation (1.1) that it is a priori clear that all the sequence $\{u_\varepsilon\}$ converges to the same limit u (see §2). But we point out that one cannot avoid this ad hoc definition by using the definition of viscosity solutions given by Lions [7], which is a direct generalization of the definition given in the case of first-order Hamilton–Jacobi equations (see [3]). The reason is that in general we would not have uniqueness in such a class of viscosity solutions (see [1]). In addition, we shall construct discontinuous "viscosity" solutions, while viscosity solutions in the sense of Lions are continuous by definition.

So let $u(x, t)$ be the viscosity solution of Problem I. In §6 we shall show that if u_0 satisfies hypotheses (H1), then

$$N = 1 \Rightarrow u \in C(\overline{Q})$$

and

$$\gamma > \tfrac{1}{2}N \Rightarrow u \in C(\overline{Q}) \cap C^{1,1}(Q).$$

It is natural to ask what the smoothness of u is if $N \geq 2$ and $0 \leq \gamma \leq N/2$.

Our first main result is that if

$$N \geq 2 \quad \text{and} \quad 0 \leq \gamma < 1$$

then u is not necessarily continuous in $\overline{Q}$. In particular, we construct a class of initial functions for which $u \notin C(\overline{Q})$. In §3 we reduce this construction to a problem concerning the Green's function for the Laplacian. The latter problem is solved in §4.

If

$$N \geq 3 \quad \text{and} \quad 1 \leq \gamma < \tfrac{1}{2}N$$

we prove a weaker result: in §5 we construct a sequence of classical and positive viscosity solutions $u_n(x, t)$ of Problem I, with u_0 replaced by u_{0_n}, which are *uniformly bounded in* Q but which are *not locally equicontinuous in* Q; i.e., there exists a compact set $G \subset Q$ such that the functions u_n are not equicontinuous on G. Of course, this does not imply that there actually exist discontinuous viscosity solutions of Problem I. On the other hand, the property that uniformly bounded solutions are locally equicontinuous is a fundamental regularity result for a large class of degenerate parabolic equations, such as the porous media equation

$$u_t = \Delta(|u|^{m-1}u), \qquad m > 1.$$

We refer the reader to [4].

About the proofs, we remark that they are based on a detailed description of, roughly speaking, the set where $u = 0$, which has been studied recently in [2].

2. Basic results

First we describe the construction of the viscosity solution $u(x, t)$ of Problem I. Let $w_\varepsilon(x, t)$ be the unique solution in $C^{2,1}(Q) \cap C(\overline{Q}) \cap L^\infty(Q)$ of

Problem I with equation (1.1) replaced by

$$u_t = u\,\Delta u - \gamma|\nabla u|^2 + \varepsilon\,\Delta u$$

where $\varepsilon > 0$. Then

$$u_\varepsilon = w_\varepsilon + \varepsilon$$

is a classical solution of Problem I with u_0 replaced by $u_{0_\varepsilon} \equiv u_0 + \varepsilon$. Since $u(x, t)$ is nonincreasing with respect to ε,

$$u(x, t) \equiv \lim_{\varepsilon \searrow 0} u_\varepsilon(x, t) = \lim_{\varepsilon \searrow 0} w_\varepsilon(x, t) \tag{2.1}$$

is well defined for all $(x, t) \in \overline{Q}$.

In §6 we shall prove that u is a weak solution of Problem I, so we arrive at the following result:

Proposition 2.1. *Let u_0 satisfy hypothesis* (H1) *and let u be defined by* (2.1). *Then u is a solution of Problem* I. *We call u the viscosity solution of Problem* I.

As we explained in the introduction, we are mainly interested in discontinuity results. However, in many cases viscosity solutions are continuous. In the next proposition we list some of the regularity results.

Proposition 2.2. *Let u_0 satisfy hypothesis* (H1), *let $\gamma \geq 0$, and let $u(x, t)$ be the viscosity solution of Problem* I. *Then*:

(i) $u_t \geq -u/t$ *and* $\Delta u \geq -1/t$ *in* $\mathscr{D}'(Q)$;
(ii) u *is continuous at points* $(x, 0)$ *for all* $x \in \mathbf{R}^N$;
(iii) *if* $u(x_0, t_0) = 0$ *for some* (x_0, t_0), *then* u *is continuous at* (x_0, t_0);
(iv) *if either* $N = 1$ *or* $\gamma > \frac{1}{2}N$, *then* $u \in C(\overline{Q})$.

The proof of (i) is given in [2]; for the proof of (ii) we refer to §6; (iii) is an immediate consequence of the following lemma.

Lemma 2.3. *Let u be the viscosity solution of Problem* I. *Then for each* $(x_0, t_0) \in \overline{Q}$

$$u(x_0, t_0) \geq \limsup_{\substack{(x, t)\to(x_0, t_0)\\ (x, t)\in\overline{Q}}} u(x, t).$$

Proof. Let $\delta > 0$ be arbitrary. From the construction of the viscosity solution it follows that it can be approximated from above by classical solutions u_ε. We fix $\varepsilon > 0$ so that $u_\varepsilon(x_0, t_0) < u(x_0, t_0) + \delta/2$. Since u_ε is continuous, there exists an open neighborhood $\mathscr{U}$ of (x_0, t_0) in $\overline{Q}$ such that

$$u_\varepsilon < u(x_0, t_0) + \delta \quad \text{in } \mathscr{U}$$

and thus

$$u < u(x_0, t_0) + \delta \quad \text{in } \mathscr{U}.$$

In the above proof we see clearly that an important property of the viscosity solution is the fact that it can be approximated from above. In general, we shall denote by $\{u_n(x, t)\}$, $n \in \mathbf{N}$, any sequence of positive classical solutions such that $u_n \searrow u$ as $n \to +\infty$ pointwise.

To complete the proof of Proposition 2.2 we observe that (iv) is a consequence of (ii) and of the following equicontinuity result.

Proposition 2.4. *Let* $\{u^i\}$, $i \in \mathbf{N}$, *be a sequence of viscosity solutions of Problem* I *such that for some* $M > 0$:

$$0 \le u^i \le M \quad in\ \overline{Q}, \ \ for\ all\ i \in \mathbf{N}.$$

(i) *If* $\gamma > N/2$, *then the functions* u^i *are uniformly Lipschitz continuous on the sets* $\mathbf{R}^N \times [\delta, +\infty]$ *for any* $\delta > 0$.

(ii) *If* $N = 1$ *and* $\gamma \ge 0$, *then the functions* u^i *are uniformly Lipschitz continuous with respect to* x *and uniformly Hölder continuous with exponent* $1/2$ *with respect to* t *on the sets* $\mathbf{R} \times [\delta, \infty]$ *for any* $\delta > 0$.

The proof of (i) follows at once from the estimates

$$-\frac{1}{t}u \le u_t \le \frac{N}{(2\gamma - N)t}u \quad \text{and} \quad |\nabla u|^2 \le \frac{2u}{(2\gamma - N)t} \quad \text{in } \mathscr{D}'(Q),$$

which are proved in [2]. To prove (ii) we use Proposition 2.2(i): if $N = 1$ then $u_{xx} \ge -1/t$; combined with the boundedness of u^i this yields at once the uniform Lipschitz continuity in x. The Hölder continuity in t follows from a result by Gilding [6].

Since equation (1.1) degenerates only at points where $u = 0$, we expect u to be smooth at points where it is positive. Since u is not continuous in general, we have to define what we mean by positivity of u.

We define the positivity set $P \subset \overline{Q}$ by

$$P = \{(x, t) \in \overline{Q} : \operatorname{ess\,inf}\{u(\xi, \tau) : (\xi, \tau) \in U\} > 0 \text{ for some neighborhood } U \text{ of } (x, t) \text{ which is open in } \overline{Q}\} \tag{2.2}$$

and set

$$P(\tau) = P \cap \{t = \tau\} \quad \text{for } \tau \ge 0.$$

In [2, Remark 2.1] it has been shown that

$$P(t) = \left\{x \in \mathbf{R}^N : \liminf_{y \to x} u(y, t) > 0\right\} \quad \text{for } t \ge 0.$$

With this definition of the positivity set we have:

Proposition 2.5 [2]. *Let* u *be the viscosity solution of Problem* I *and let the open set* $P \subset \overline{Q}$ *be defined by* (2.2). *Then* $u \in C^{2,1}(P \cap Q) \cap C(P)$ *and* u *satisfies* (1.1) *in* $P \cap Q$.

In [2] a detailed description of P was given. In particular:

$$\overline{P(t)} = \overline{P(0)} \quad \text{for all } t \ge 0 \tag{2.3}$$

and

$$P(t_1) \subset P(t_2) \quad \text{if } 0 \le t_1 < t_2. \tag{2.4}$$

The set P is not always completely determined by (2.3), for example, when u_0 has an isolated zero. To illustrate this we assume that u_0 satisfies, in addition to (H1), the following hypothesis.

$$\text{(H2)} \qquad u_0(0) = 0 \quad \text{and} \quad u_0 > 0 \quad \text{in } \mathbf{R}^N \backslash \{0\}.$$

Then it is not clear what happens at $x = 0$. Defining $t^* \geq 0$ by

$$t^* = \sup\{t \geq 0 : 0 \notin P(t)\}, \tag{2.5}$$

it follows from (2.4) that

$$0 \notin P(t) \quad \text{for } 0 \leq t < t^*,$$

and

$$0 \in P(t) \quad \text{for } t > t^*.$$

We call t^* the *waiting time* at $x = 0$. It turns out that t^* depends heavily on γ, N, and the local behavior of u near $x = 0$. In particular, t^* can be zero, nonzero and finite, and infinite. For precise results we refer to [2].

3. Examples of discontinuous viscosity solutions if $0 \leq \gamma < 1$, $N \geq 2$

In this section we shall show that for some class of initial data the viscosity solution is not continuous if $N \geq 2$ and $0 \leq \gamma < 1$. To be precise, we shall prove the following theorem.

Theorem 3.1. *If $0 \leq \gamma < 1$ and $N \geq 2$, then there exists a family of initial data satisfying hypotheses* (H1) *and* (H2) *such that the corresponding viscosity solutions of Problem* I *cannot be redefined on a set of measure zero as continuous functions in $\overline{Q}$.*

In the proof of the theorem we shall give a clear picture of the discontinuity of u. In particular, we construct a class of initial functions such that there exists a time interval of positive measure in which

$$u(0, t) > 0 \quad \text{and} \quad \liminf_{x \to 0} u(x, t) = 0.$$

The construction is based on some properties of the Green's function for the Laplacian and on a result concerning the waiting time t^* at $x = 0$ in [2]. To complete the proof we shall show that in the same time interval

$$\limsup_{x \to 0} u(x, t) = u(0, t) > \liminf_{x \to 0} u(x, t).$$

Since u is continuous for $x \neq 0$ (cf. Proposition 2.5), this implies that u cannot be redefined as a continuous function in $\overline{Q}$.

First we introduce some notation. Let u_0 satisfy assumptions (H1) and (H2) and define for $x \neq 0$

$$f(x) = \begin{cases} u_0^{-\gamma}(x) & \text{if } 0 < \gamma < 1, \\ |\log u_0(x)| & \text{if } \gamma = 0. \end{cases} \tag{3.1}$$

Let $\rho > 0$ be fixed, let $B_\rho(y) = \{x \in \mathbf{R}^N : |x-y| < \rho\}$, and define for $y \in \mathbf{R}^N$

$$I(y) = \int_{B_\rho(y)} G(|x-y|)f(x)\,dx, \tag{3.2}$$

where G is the Green's function for the Laplacian in $B_\rho(0)$ defined by

$$G(r) = \begin{cases} r^{2-N} - \rho^{2-N} & \text{if } N \geq 3, \\ \log(\rho/r) & \text{if } N = 2, \end{cases} \tag{3.3}$$

for $r > 0$.

We are interested in initial functions which satisfy one of the two following conditions.

(H3) Let $I(y)$ be defined by (3.2). Then

$$I(y) < \infty \quad \text{for all } y \in \mathbf{R}^N$$

and

$$\limsup_{y\to 0} I(y) = \infty.$$

(H4) Let $I(y)$ be defined by (3.2). Then I is uniformly bounded in $\mathbf{R}^N$ and discontinuous at $y = 0$.

The following result implies immediately that there do exist initial functions u_{01} and u_{02} which satisfy, in addition to (H1) and (H2), hypotheses (H3) and (H4), respectively.

Lemma 3.2. *Let $N \geq 2$ and $0 \leq \gamma < 1$. Define*

$$\mathscr{F} = \Big\{ f : \mathbf{R}^N \backslash \{0\} \to \mathbf{R}^+,\ f \in C(\mathbf{R}^N\backslash\{0\}),\ \lim_{x\to 0} f(x) = \infty, \\ \textit{and } f \geq \mathscr{C} \textit{ in } \mathbf{R}^N\backslash\{0\} \textit{ for some } \mathscr{C} > 0 \Big\}.$$

Let $I(y)$ be defined by (3.2). Then the two following sets are nonempty:

$$\mathscr{F}_1 = \{f \in \mathscr{F} : f \textit{ satisfies } \mathrm{H3}\},$$
$$\mathscr{F}_2 = \{f \in \mathscr{F} : f \textit{ satisfies } \mathrm{H4}\}.$$

Since we are not aware of a proof of this result in the literature, we shall give the proof in §4.

Let u_0 satisfy (H1) and (H2). Let the waiting time t^* at $x = 0$ be defined by (2.5). Then, by [2, Theorem 3.2(iii)],

$$t^* = \infty \quad \text{if } u_0 \text{ satisfies (H3)}, \tag{3.4}$$

and

$$t^* < \infty \quad \text{if } u_0 \text{ satisfies (H4)}. \tag{3.5}$$

Here we recall that

$$\liminf_{y\to 0} u(y, t) = 0 \quad \text{if } t < t^*.$$

On the other hand, we can prove the following result.

Lemma 3.3. *Let $N \geq 2$ and $0 \leq \gamma < 1$ and let u_0 satisfy* (H1) *and* (H2). *If $I(0) < \infty$, then there exists $0 \leq t_0 < \infty$ such that*

$$u(0, t) \begin{cases} = 0 & \text{if } 0 \leq t < t_0, \\ > 0 & \text{if } t > t_0. \end{cases}$$

The proof is quite similar to the one of Theorem 2.2(iii) in [2], and we omit it here.

Now we arrive at our main result, which says that initial functions which satisfy (H3) or (H4) yield the discontinuous viscosity solutions of which we claimed the existence in Theorem 5.1.

Proposition 3.4. *Let $0 \leq \gamma < 1$ and $N \geq 2$ and let u_0 satisfy the hypotheses* (H1), (H2), *and either* (H3) *or* (H4). *Let $t_0 \geq 0$ be defined by Lemma* 3.3 *and let t^* be the waiting time at $x = 0$, satisfying* (3.4), *respectively* (3.5). *Then $t_0 < t^*$ and the viscosity solution u of Problem* I *satisfies*

$$u \in C(\overline{Q} \backslash \{\{0\} \times [t_0, t^*]\}), \tag{3.6}$$

and, for all $t_0 < t < t^*$,

$$\limsup_{x \to 0} u(x, t) > 0 \quad \text{and} \quad \liminf_{x \to 0} u(x, t) = 0. \tag{3.7}$$

In particular, u cannot be made continuous in Q by redefining it on a set of measure zero.

Proof. First we prove (3.6). Since $u_0(x) > 0$ if $x \neq 0$, u is positive and continuous in a point (x, t) if $x \neq 0$. By Lemma 3.3 and Proposition 2.2(iii), u is also continuous in the points $(0, t)$ if $t < t_0$. If $t^* < \infty$ and $t > t^*$, then $0 \in P(t)$. Hence, by Proposition 2.5, u is continuous at $(0, t)$ if $t > t^*$, and (3.6) follows.

Next we show that $t_0 < t^*$. If u_0 satisfies (H3) this is obvious, since $t_0 < \infty$ and $t^* = \infty$.

So let u_0 satisfy (H4). Clearly, $t_0 \leq t^* < \infty$. Thus, arguing by contradiction, we may suppose that $t_0 = t^*$.

Let $0 < \gamma < 1$. Following the proof of Theorem 3.2(iii) in [2], it follows that for certain positive constants a and b and for all $y \in \mathbf{R}^N$ and $t > t^*$

(3.8)
$$\int_0^t u^{1-\gamma}(y, \tau)\, d\tau = a \int_0^t \int_{\partial B_\rho(y)} u^{1-\gamma}\, dx\, d\tau + \frac{b(1-\gamma)}{\gamma} \int_{B_\rho(y)} G(|x-y|) u^{-\gamma}(x, t)\, dx - \frac{b(1-\gamma)}{\gamma} I(y).$$

By hypothesis (H4), $I(y)$ is discontinuous at $y = 0$. The first and second terms on the right-hand side of (3.8) are continuous at $y = 0$. Concerning the second term, this follows from the fact that, since $t > t^*$, $u^{-\gamma}$ is bounded near $(0, t)$. Hence

$$\int_0^t u^{1-\gamma}(y, \tau)\, d\tau \quad \text{is discontinuous at } y = 0, \ \text{ for } t > t^*.$$

On the other hand, since $t_0 = t^*$, it follows from (3.6) that

$$\int_0^t u^{1-\gamma}(y, \tau)\,d\tau \quad \text{is continuous at } y = 0, \ \text{ for all } t \geq 0,$$

and we have found a contradiction.

If $\gamma = 0$, the proof is similar: we replace the last two terms in (3.8) by

$$-b \int_{B_\rho(y)} G(|x-y|) \log u(x, t)\,dx + bI(y)$$

and proceed as before.

Finally, we prove (3.7). By the definition of t^*

$$\liminf_{x \to 0} u(x, t) = 0 \quad \text{for all } t < t^*,$$

so it remains to prove that

$$\limsup_{x \to 0} u(x, t) > 0 \quad \text{for all } t > t_0.$$

This is a consequence of Lemma 3.3 and the following result.

Lemma 3.5. *Let* u_0 *satisfy assumptions* (H1) *and* (H2). *Then*

$$\limsup_{x \to 0} u(x, t) = u(0, t) \quad \textit{for all } t \geq 0.$$

Proof. By Lemma 2.3

$$u(0, t) \geq \limsup_{x \to 0} u(x, t) \quad \text{for } t \geq 0.$$

We argue by contradiction. Let $\delta > 0$ be arbitrary and fix $t \geq \delta$. Suppose that

$$\limsup_{x \to 0} u(x, t) = c < u(0, t).$$

Then by definition and since u is continuous for $x \neq 0$, for any fixed $\varepsilon > 0$ such that $c + 2\varepsilon < u(0, t)$, there exists a $\rho > 0$ such that

$$u(x, t) \leq c + \varepsilon \quad \text{for } x \in \overline{B_\rho(0)} \backslash \{0\}.$$

Since $u(x, t) > 0$ for $x \in \partial B_\rho(0)$, the nonincreasing approximating sequence u_n converges uniformly to u on $\partial B_\rho(0) \times \{t\}$. Therefore we have for u sufficiently large

$$u_n(x, t) \leq c + 2\varepsilon \quad \text{for } x \in \partial B_\rho(0).$$

On the other hand, we know from Proposition 2.2(i) that

$$\Delta(u_n + (1/2\delta)|x|^2) \geq 0 \quad \text{for } t \geq \delta > 0.$$

Therefore, by the maximum principle, u_n cannot have interior maxima in $B_\rho(0)$, i.e.,

$$u_n(0, t) \leq c + 2\varepsilon < u(0, t).$$

Hence we get a contradiction, since $u_n(0, t) \searrow u(0, t)$ as $n \to \infty$.

4. Proof of Lemma 3.2

Throughout this section we shall use the same notation as in §2.

We shall construct the functions f by modifying the radially symmetric function $1+|x|^{-\alpha}$ $(0<\alpha<2)$ on neighborhoods of a suitable sequence of points $x_n \to 0$.

Let $\{x_n\}$ be a sequence such that

$$x_n \to 0 \text{ as } n\to\infty, \qquad x_n \in B_{1/2}(0)\backslash\{0\}, \quad x_n \neq x_m \text{ if } n \neq m. \tag{4.1}$$

We choose a sequence $\{\rho_n\}$ such that

$$0<\rho_n \to 0 \text{ as } n \to \infty, \qquad B_n \cap B_m = \varnothing \text{ if } n\neq m \text{ and } 0 \notin B_n \text{ for all } n, \tag{4.2}$$

where we have written $B_n = B_{\rho_n}(x_n)$.

Let v be a function on $\mathbf{R}^N$ such that:

(4.3) v is radially symmetric, smooth, nonnegative, decreasing with respect to $r=|x|$, $\operatorname{supp} v \subset B_1(0)$, and $\int_{\mathbf{R}^N} v(x)\,dx = 1$.

For any $j \in \mathbf{N}$ define

$$v_j^n(x) = j^N v(j(x-x_n)). \tag{4.4}$$

From (4.3) we then have for any j, n:

$$\int_{\mathbf{R}^N} v_j^n(x)\,dx = 1. \tag{4.5}$$

Observe that, for n fixed, v_j^n approximates $\delta(x-x_n)$ as $j\to+\infty$.

By the definition (4.3) of $v(x)$, for any fixed n there exists a $\hat{j_n}$ such that

$$\operatorname{supp} v_j^n \subset B_{\rho_n/2}(x_n) \subset B_n \quad \text{for any } j \geq \hat{j_n}.$$

For any n let $j_n \geq \hat{j_n}$ be chosen later.

First let $N\geq 3$. For any $x \in \mathbf{R}^N\backslash\{0\}$ we define:

$$f(x) = 1+|x|^{-\alpha} + \sum_{n=1}^{\infty} v_{jn}^n(x)|x-x_n|^\beta \tag{4.6}$$

with $0<\beta\leq N-2$, $0<\alpha<2$.

Claim 4.1. Let $N\geq 3$ and f be defined in (4.6). Then we can choose j_n such that:

(i) if $0<\beta<N-2$ then $f\in\mathscr{F}_1$;
(ii) if $\beta = N-2$ then $f \in \mathscr{F}_2$.

If $N=2$ we define for any $x\in\mathbf{R}^2\backslash\{0\}$:

$$f(x) = \begin{cases} 1+|x|^{-\alpha} + \sum_{n=1}^\infty v_{j_n}^n(x)(-\log|x-x_n|)^{-\sigma} & \text{for } x\neq x_n, \\ 1+|x|^{-\alpha} & \text{for } x = x_n, \end{cases} \tag{4.7}$$

with $0<\sigma\leq 1$, $0<\alpha<2$.

Claim 4.2. Let $N = 2$ and let f be defined by (4.7). Then we can choose j_n such that:

(i) if $0 < \sigma < 1$ then $f \in \mathscr{F}_1$;
(ii) if $\sigma = 1$ then $f \in \mathscr{F}_2$.

Clearly, the proof of these claims completes the proof of Lemma 3.2. Here we shall give only the proof of Claim 4.1 since the one of Claim 4.2 is quite similar.

Proof of Claim 4.1. Since the supports of $v_{j_n}^n$ are mutually disjoint, it is easy to see that $f \in \mathscr{F}$. We consider the function $I(y)$, defined by (3.2).

Clearly, it is enough to study $I(y)$ for $y \in B_1(0)$. Without loss of generality we may assume that $\rho > 1$. Therefore it is sufficient to study the function

$$g(y) = \int_{B_1(0)} \overline{G}(|x-y|) f(x)\,dx \tag{4.8}$$

where

$$\overline{G}(r) = \begin{cases} r^{2-N} & \text{if } N \geq 3, \\ -\log r & \text{if } N = 2, \end{cases} \tag{4.9}$$

for $r > 0$.

We first state a result about the first part $1 + |x|^{-\alpha}$ of the function f.

Lemma 4.3. *Let*

$$w(y) = \int_{B_1(0)} \overline{G}(|x-y|)(1 + |x|^{-\alpha})\,dx \quad \textit{for } y \in \mathbf{R}^N. \tag{4.10}$$

If $0 < \alpha < 2$, *then* $w \in C(\mathbf{R}^N)$.

The proof is straightforward and we leave it to the reader.

We continue the proof of Claim 4.1. First we have to check that $g(y) < \infty$ for any $y \in \mathbf{R}^N$. Let

$$A_n(y) = \int_{B_n} G(|x-y|) v_{j_n}^n(x) |x - x_n|^\beta\,dx. \tag{4.11}$$

Recalling that the supports of $v_{j_i}^i$ are mutually disjoint, we derive, using the monotone convergence theorem, that

$$\begin{aligned} g(y) &= w(y) + \int_{B_1(0)} \overline{G}(|x-y|) \sum_{k=1}^{\infty} (v_{j_k}^k(x) |x - x_k|^\beta)\,dx \\ &= w(y) + \int_{B_1(0)} \overline{G}(|x-y|) \lim_{L \to +\infty} \left(\sum_{k=1}^{L} v_{j_k}^k(x) |x - x_k|^\beta \right) dx \\ &= w(y) + \lim_{L \to +\infty} \int_{B_1(0)} \overline{G}(|x-y|) \sum_{k=1}^{L} (v_{j_k}^k(x) |x - x_k|^\beta)\,dx \\ &= w(y) + \sum_{n=1}^{\infty} A_n(y) < \infty \quad \text{for } y \in \mathbf{R}^N \end{aligned} \tag{4.12}$$

provided

$$\sum_{n=1}^{\infty} A_n(y) < \infty \quad \text{for } y \in \mathbf{R}^N .$$

Lemma 4.4. *We can choose j_n such that*

$$\sum_{n=1}^{\infty} A_n(y) < \infty \quad \textit{for any } y \in \mathbf{R}^N .$$

Proof. First let $y \notin \bigcup_{n=1}^{\infty} B_n$. For any n

$$\sup_{\substack{y \notin B_n \\ x \in \operatorname{supp} v_{j_n}^n}} |x-y|^{2-N} \le C_n .$$

By (4.3)

$$A_n(y) \le C_n \int_{B_n} v_{j_n}^n(x)|x-x_n|^{\beta}\, dx \le C_n j_n^{-\beta} \quad \text{for } y \notin B_n .$$

Hence for any given positive ε_n we can choose j_n so large that

$$A_n(y) < \varepsilon_n \quad \text{for } y \notin B_n .$$

Choose a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\sum_{n=1}^{\infty} \varepsilon_n \le \tfrac{1}{2} .$$

Then

$$\sum_{n=1}^{\infty} A_n(y) \le \tfrac{1}{2} \quad \text{for } y \notin \bigcup_{n=1}^{\infty} B_n . \tag{4.13}$$

Now consider the case $y \in \bigcup_{n=1}^{\infty} B_n$. Since the B_n are mutually disjoint, every fixed y belongs to at most one of the balls, say B_k. Then for any $L > k$

$$\sum_{n=1}^{L} A_n(y) = \sum_{\substack{n=1 \\ n \ne k}}^{L} A_n(y) + A_k(y) .$$

Since, for any fixed k, $A_k(y) < \infty$ for $y \in \mathbf{R}^{\mathrm{N}}$, we have, as before, that we can choose j_n so that

$$\sum_{n=1}^{\infty} A_n(y) \le \tfrac{1}{2} + A_k(y) < \infty \quad \text{for } y \in B_k . \tag{4.14}$$

This completes the proof of Lemma 4.4.

Next consider:

$$\begin{aligned} A_n(x_n) &= \int_{B_n} |x-x_n|^{2-N+\beta} v_{j_n}^n(x)\, dx \\ &= j_n^{\mathrm{N}-2-\beta} \int_{B_1(0)} |z|^{2-\mathrm{N}+\beta} v(z)\, dz . \end{aligned} \tag{4.15}$$

Let $0 < \beta < N-2$. Then, by (4.12) and (4.15),

$$g(x_n) = w(x_n) + \sum_{k=1}^{\infty} A_k(x_n) \geq A_n(x_n) \to +\infty \quad \text{as } n \to +\infty,$$

and hence part (i) of Claim 4.1 follows.

If $\beta = N-2$ it is easy to show, by the definition of $A_n(y)$ and by (4.3), that there exists a positive constant C, independent of n, such that

$$A_n(y) < C \quad \text{for any } y \in \mathbf{R}^N, \quad n \in \mathbf{N}. \tag{4.16}$$

Moreover, by (4.15)

$$A_n(x_n) = 1 \quad \text{for any } n \in \mathbf{N}.$$

Therefore, by (4.12), (4.13), (4.14), and (4.16), $g(y)$ is uniformly bounded in $\mathbf{R}^N$ and

$$g(x_n) \geq w(x_n) + 1 \quad \text{for any } n \in \mathbf{N}. \tag{4.17}$$

We argue by contradiction and suppose that g is continuous in $\mathbf{R}^N$. By Lemma 4.3, w is continuous, and hence (4.17) implies that

$$g(0) \geq w(0) + 1.$$

On the other hand, since $0 \notin \bigcup_{n=1}^{\infty} B_n$, we obtain from (4.13) that

$$g(0) = w(0) + \sum_{n=1}^{\infty} A_n(0) \leq w(0) + \tfrac{1}{2}$$

and we have a contradiction. So also part (ii) of Claim 4.1 has been proved.

5. The case $1 \leq \gamma < N/2$, $N \geq 3$: A counterexample of equicontinuity

If $1 \leq \gamma < N/2$ and $N \geq 3$, it is an open Problem whether there exist discontinuous viscosity solutions of Problem I. Instead, we prove the following result.

Theorem 5.1. *Let $N \geq 3$ and $1 \leq \gamma < N/2$. Then there exists a sequence $\{\tilde{u}_j\}_{j=1,2,\dots} \subset C^{2,1}(B_1 \times \overline{\mathbf{R}^+})$ such that $\tilde{u}_j$ satisfies equation* (1.1) *in $B_1 \times \overline{\mathbf{R}^+}$,*

$$0 < \tilde{u}_j \leq 2 \quad \text{in } B_1 \times \overline{\mathbf{R}^+},$$

but the functions $\tilde{u}_j$ are not equicontinuous in $B_{1/2} \times \overline{\mathbf{R}^+}$.

Here B_r denotes the ball $B_r(0)$.

Remarks. (i) For simplicity, we do not construct solutions u_j on all of $\mathbf{R}^N \times \overline{\mathbf{R}^+}$ but only on $B_1 \times \overline{\mathbf{R}^+}$.

(ii) Observe that strictly positive solutions of Problem I are viscosity solutions. In that sense the construction of a sequence of strictly positive solutions u_j is the strongest possible counterexample to local equicontinuity.

Proof of Theorem 5.1. First let $1 < \gamma < N/2$.

We consider the Dirichlet problem

$$\text{(II)} \qquad \begin{cases} u_t = u\,\Delta u - \gamma|\nabla u|^2 & \text{in } \mathscr{Q}_1 \equiv B_1 \times \mathbf{R}^+, \\ u = 1 & \text{in } \partial B_1 \times \mathbf{R}^+, \\ u(x, 0) = u_0(x) \equiv |x|^\alpha & \text{for } x \in B_1, \end{cases}$$

for some $\alpha > 2$ to be chosen later.

Again, Problem II has a uniquely determined viscosity solution $u(x, t)$, which can be approximated from above by classical solutions u_n of the problem.

$$\text{(II}_n\text{)} \qquad \begin{cases} u_t = u\Delta u - \gamma|\nabla u|^2 & \text{in } \mathscr{Q}_1, \\ u = 1 + (1/n) & \text{in } \partial B_1 \times \mathbf{R}^+, \\ u(x, 0) = u_{0n}(x) \equiv |x|^\alpha + (1/n) & \text{for } x \in B_1. \end{cases}$$

Lemma 5.2. *Let* u *be the viscosity solution of Problem* II. *Then* $u \in C^{2,1}(\{\overline{B}_1\backslash\{0\} \times [0, \infty))$ *and* $u(0, t) = \lim_{x\to 0} u(x, t)$ *for* $t \geq 0$.

Proof. By (2.4) and Proposition 2.5,

$$u > 0 \text{ and } u \text{ is smooth in } \{\overline{B}_1\backslash\{0\}\} \times [0, \infty).$$

By standard arguments

$$u_n(x, t) = u_n(r, t), \qquad r = |x|,$$

and

$$\partial u_n/\partial r > 0 \quad \text{if } 0 < r \leq 1, \ t \geq 0. \tag{5.2}$$

Since $u_n \searrow u$ pointwise as $n \to \infty$, this, together with Lemma 2.3, implies that $u(0, t) = \lim_{x\to 0} u(x, t)$.

We now arrive at the heart of the proof of Theorem 5.1. The following result shows that it is possible to choose α in such a way that u is not uniformly continuous in $\mathscr{Q}_1$. Later we shall use this to construct the desired sequence.

Proposition 5.3. *Let* $1 < \gamma < N/2$ *and* $N \geq 3$. *Let* u *be the viscosity solution of Problem* II. *If*

$$2 < \alpha < (N-2)/(\gamma - 1), \tag{5.3}$$

then

(i) $u(0, t) = 0$ *for* $t \geq 0$,

(ii) $u(x, t)$ *is nondecreasing with respect to* t, *and*

$$u(\cdot, t) \nearrow 1 \quad \textit{uniformly on compact subsets of } \overline{B}_1\backslash\{0\} \textit{ as } t \to \infty. \tag{5.4}$$

Proof. (i) Let the waiting time t^* at $x = 0$ be defined by (2.5). Since $\alpha > 2$ it follows from [2, Figure 2] that $t^* = \infty$. By Lemma 5.2 this implies that $u(0, t) = 0$ for $t \geq 0$.

(ii) To prove that u is nondecreasing in t, it is sufficient to show that $u_{nt} \geq 0$ in $\mathscr{Q}_1$. This follows from standard monotonicity methods provided we show that u_{0n} is a subsolution of the associated steady-state problem, i.e., that

$$u_{0n}\Delta u_{0n} - \gamma|\nabla u_{0n}|^2 = u_0\Delta u_0 - \gamma|\nabla u_0|^2 + (1/n)\Delta u_0 \geq 0 \quad \text{in } B_1 .$$

To see this we compute

$$\Delta u_0 = \alpha(\alpha + N - 2)|x|^{\alpha-2} \geq 0 \quad \text{in } B_1$$

and

$$u_0\Delta u_0 - \gamma|\nabla u_0|^2 = \alpha\{N - 2 - (\gamma - 1)\alpha\}|x|^{2\alpha-2} \geq 0 \quad \text{in } B_1 ,$$

where we used that $\alpha < (N-2)/(\gamma-1)$.

To prove (5.4) we have to show that for fixed $0 < \rho < 1$

$$u(\cdot, t) \nearrow 1 \quad \text{uniformly on } \overline{B}_1\backslash B_\rho \text{ as } t \to \infty . \tag{5.5}$$

Let $\sigma \in (0, \rho)$ be chosen. Let v be the unique solution of the nondegenerate problem

$$\text{(III)} \qquad \begin{cases} v_t = v\,\Delta v - \gamma|\nabla v|^2 & \text{in } \{B_1\backslash\overline{B_\sigma}\} \times \mathbf{R}^+ , \\ v = 1 & \text{in } \partial B_1 \times \mathbf{R}^+ , \\ v = \sigma^\alpha & \text{in } \partial B_\sigma \times \mathbf{R}^+ , \\ v(x, 0) = |x|^\alpha & \text{in } B_1\backslash B_\sigma . \end{cases}$$

Since u is nondecreasing in t, $u \geq u_0 = \sigma^\alpha$ on $\partial B_\sigma \times \mathbf{R}^+$, and thus, by the classical comparison principle,

$$v \leq u \leq 1 \quad \text{in } \overline{B\backslash B_\sigma} \times [0, \infty) . \tag{5.6}$$

We are interested in the steady-state solutions of Problem III. Since $\sigma^\alpha \leq v \leq 1$ in $\{B_1\backslash B_\sigma\} \times \mathbf{R}^+$ and v satisfies

$$(v^{-\gamma})_t = \frac{\gamma}{\gamma - 1}\operatorname{div}(v^{1-\gamma}) ,$$

we arrive at the steady-state problem

$$\text{(IV)} \qquad \begin{cases} \Delta(w^{1-\gamma}) = 0 & \text{in } B\backslash B_\sigma , \\ w = 1 & \text{on } \partial B_1 ,\ w = \sigma^\alpha \text{ on } \partial B_\sigma , \\ 0 < \sigma^\alpha \leq w \leq 1 & \text{in } B\backslash\overline{B_\sigma} . \end{cases}$$

Clearly, Problem IV has a unique solution $w_\sigma \in C^2(\overline{B\backslash B_\sigma})$ and, by standard arguments,

$$v(\cdot, t) \to w_\sigma \quad \text{uniformly on } B\backslash B_\sigma \text{ as } t \to \infty . \tag{5.7}$$

In view of (5.6) and (5.7), the proof of (5.5) (and hence of Proposition 5.3) is completed by the following lemma.

Lemma 5.4. *Let* $\alpha < (N-2)/(\gamma-1)$, *let* $0 < \sigma < 1$, *and let* w_σ *be the unique solution of Problem* IV. *Then* $w_\sigma \nearrow 1$ *uniformly on compact subsets of* $\overline{B_1}\backslash\{0\}$ *as* $\sigma \searrow 0$.

Proof. An explicit calculation shows that

$$w_\sigma^{-(\gamma-1)}(x) = 1 + \delta(\sigma)\{|x|^{-(N-2)} - 1\}$$

where

$$\delta(\sigma) = (\sigma^{-(\gamma-1)\alpha} - 1)/(\sigma^{-(N-2)} - 1).$$

The proof is completed by the observation that $\delta(\sigma) \searrow 0$ as $\sigma \searrow 0$.

We continue the proof of Theorem 5.1. Let $\rho_i \searrow 0$ as $i \to \infty$. By (5.4) we can choose $T_i > 0$ such that

$$|u(x, t) - 1| \le 1/i \quad \text{for } x \in B_1\backslash B_{\rho_i}, \ t \ge T_i. \tag{5.8}$$

Next we choose n_i such that

$$u_{n_i}(0, T_i + 1) \le u(0, T_i + 1) + 1/i = 1/i, \tag{5.9}$$

where we have used Proposition 5.3(i).

We define

$$\tilde{u}_i(x, t) = u_{n_i}(x, t + T_i), \qquad x \in \overline{B}, \ t \ge 0.$$

Since $u_{nt} \ge 0$ it follows from (5.9) that

$$\tilde{u}_i(0, t) \le u_{n_i}(0, T_i + 1) \le 1/i \quad \text{if } 0 \le t \le 1$$

and hence

$$\tilde{u}_i(0, t) \to 0 \quad \text{as } i \to \infty, \ \text{uniformly in } t \in [0, 1]. \tag{5.10}$$

On the other hand, it follows from (5.8) that

$$\tilde{u}_i(x, t) - 1 \ge u(x, T_i + t) - 1 \ge -1/i \quad \text{for } x \in B_1\backslash B_{\rho_i}, \ t \ge 0.$$

In addition, $u_n \le 1 + 1/n$ in $\mathcal{Q}_1$ and hence

$$\tilde{u}_i(x, t) - 1 \le 1/n_i \quad \text{in } \mathcal{Q}_1,$$

and so $\tilde{u}_i \to 1$ uniformly on compact sets of $\{\overline{B}_1\backslash\{0\}\} \times [0, \infty)$. Combining this with (5.10), it follows that the functions $\tilde{u}_i$ are not equicontinuous on $B_1 \times [0, 1]$, and we have proved Theorem 5.1 in the case that $1 < \gamma < \frac{1}{2}N$.

If $\gamma = 1$ the proof is almost identical. Only condition (5.3) has to be changed to $\alpha > 2/\gamma$, and in Problem IV the role of $w^{1-\gamma}$ is now played by $-\log w$.

6. Existence and continuity at $t = 0$

In this section we prove the existence and the continuity at $t = 0$ of the viscosity solution of Problem I; i.e., we prove Propositions 2.1 and 2.2(ii).

The continuity at $t = 0$ follows immediately from the following lemma. It can be considered as a local equicontinuity result at $t = 0$ and is, for later use, slightly more general than we need right now.

Lemma 6.1. *Let u_0 satisfy* (H1) *and let $\{u_n^{(1)}\}$, $n = 1, 2, \ldots,$ and $\{u_n^{(2)}\}$, $n = 1, 2, \ldots,$ be two sequences of continuous viscosity solutions of Problem* I *such that for any n*

$$(6.1) \qquad \begin{aligned} 0 \le u_{n+1}^{(2)} \le u_n^{(2)} \le M \quad in\ \overline{Q}, \\ 0 \le u_n^{(1)} \le u_n^{(2)} \le M \quad in\ \overline{Q}, \end{aligned}$$

and such that for $i = 1, 2$

$$u_n^{(i)}(x, 0) \to u_0(x) \ in\ L^\infty_{\mathrm{loc}}(\mathbf{R}^N) \ as\ n \to \infty.$$

Let $\Omega \subset \mathbf{R}^N$ be compact. Then for any $\varepsilon > 0$ there exist $\delta_\varepsilon, n_\varepsilon, t_\varepsilon > 0$ such that for any $x_0 \in \Omega$, $n \ge n_\varepsilon$, and $i = 1, 2$

$$(6.2) \qquad |u_n^{(i)}(x, t) - u_0(x_0)| < \varepsilon \quad for\ x \in B_{\delta_\varepsilon}(x_0)\ and\ 0 \le t < t_\varepsilon.$$

Proof. By the uniform continuity of u_0 on bounded subsets of $\mathbf{R}^N$ there exists for all $\varepsilon > 0$ a $\delta_\varepsilon > 0$ such that for all $x_0 \in \Omega$

$$|u_0(x) - u_0(x_0)| < \tfrac{1}{3}\varepsilon \quad \text{for } x \in B_{2\delta_\varepsilon}(x_0).$$

Since $u_n^{(i)}(\cdot, 0) \to u_0$ uniformly on bounded sets as $n \to \infty$, there exists an n_ε such that for $x_0 \in \Omega$ and $i = 1, 2$

$$(6.3) \qquad |u_n^{(i)}(x, 0) - u_0(x_0)| < \tfrac{2}{3}\varepsilon \quad \text{for } x \in B_{2\delta_\varepsilon}(x_0) \text{ and } n \ge n_\varepsilon.$$

By the continuity of $u_{n_\varepsilon}^{(2)}$ there exists a $t_\varepsilon > 0$ such that for $x_0 \in \Omega$

$$u_{n_\varepsilon}^{2}(x, t) < u_0(x_0) + \varepsilon \quad \text{for } x \in B_{\delta_\varepsilon}(x_0) \text{ and } 0 \le t < t_\varepsilon.$$

By (6.1) and the comparison principle

$$(6.4) \quad u_n^{(i)}(x, t) < u_0(x_0) + \varepsilon \quad \text{for } x \in B_{\delta_\varepsilon}(x_0),\ 0 \le t < t_\varepsilon,\ n \ge n_\varepsilon,\ i = 1, 2.$$

It remains to prove a lower bound for $u_n^{(i)}$. Let $\varepsilon > 0$ and $x_0 \in \Omega$. Without loss of generality we may assume that $u_0(x_0) \ge \varepsilon$. Then, by (6.3),

$$(6.5) \qquad u_n^{(1)}(x, 0) > u_0(x_0) - \tfrac{2}{3}\varepsilon \ge \tfrac{1}{3}\varepsilon \quad \text{if } x \in B_{2\delta}(x_0),\ n \ge n_\varepsilon.$$

We shall compare $u_n^{(1)}$ with an explicit continuous solution. By [2, Proposition 2.4] there exists for all $\gamma \ge 0$ and $\rho > 0$ a nonincreasing function $g_\rho \in C(\overline{\mathbf{R}^+})$ such that

$$g_\rho(r) \begin{cases} > 0 & \text{if } 0 \le r < \rho, \\ = 0 & \text{if } r \ge \rho, \end{cases}$$

and such that for any $\tau > 0$ the function

$$(6.6) \qquad U(x, t) = \frac{1}{t + \tau} g_\rho(|x|)$$

is a continuous solution of Problem I. We set $\rho = \delta_\varepsilon$ and choose $\tau_\varepsilon > 0$ so that

$$g_{\delta_\varepsilon}(0)/\tau_\varepsilon = u_0(x_0) - \tfrac{2}{3}\varepsilon .$$

Then, by (6.5), (6.6), and the comparison principle, for any $y \in B_{\delta_\varepsilon}(x_0)$ and $n \geq n_\varepsilon$

$$u_n^{(1)}(x, t) \geq \frac{1}{t + \tau_\varepsilon} g_{\delta_\varepsilon}(|x - y|) \quad \text{for } (x, t) \in \overline{Q}$$

and hence, choosing $t_\varepsilon > 0$ so small that

$$g_{\delta_\varepsilon}(0)/(t_\varepsilon + \tau_\varepsilon) > u_0(x_0) - \varepsilon ,$$

we find that for all x_0 and $n \geq n_\varepsilon$

$$u_n^{(2)}(x, t) \geq u_n^{(1)}(x, t) > u_0(x_0) - \varepsilon \quad \text{if } x \in B_{\delta_\varepsilon}(x_0), \ 0 \leq t < t_\varepsilon .$$

Combining this with (6.4), the proof is complete.

Finally, we prove the existence of the viscosity solution $u(x, t)$ of Problem I, following the procedure which we described in §2.

So let $\{u_n\}_{n=1,2,\dots} \subset C^{2,1}(\overline{Q})$ be a nonincreasing sequence of bounded positive solutions of Problem I and let

$$u_n(x, t) \searrow u(x, t) \quad \text{pointwise in } \overline{Q} \text{ as } n \to \infty .$$

In view of the definition of a solution, it is enough to prove that

$$u \in L^2_{\text{loc}}([0, \infty); H^1_{\text{loc}}(\mathbf{R}^N))$$

and

$$\iint_K ||\nabla u_n|^2 - |\nabla u|^2|\, dx\, dt \to 0 \quad \text{as } n \to \infty \tag{6.7}$$

for any set $K = \Omega \times (0, T)$ with $\Omega \subset \mathbf{R}^N$ bounded.

The proof of (6.7) is a modification of the proof of a similar result in [5]. The basic estimate is given in the following lemma.

Lemma 6.2. *For any bounded cylinder $K = \Omega \times (0, T)$ and $0 \leq \alpha < 1$ there exists a constant $C_{K,\alpha}$ such that for all $n = 1, 2, \dots$*

$$\iint_K u_n^{-\alpha} |\nabla u_n|^2 \leq C_{K,\alpha} . \tag{6.8}$$

In addition, there exist for all $\varepsilon > 0$ constants $n_\varepsilon, t_\varepsilon > 0$ such that for any $n \geq n_\varepsilon$

$$\int_0^{t_\varepsilon} \int_\Omega |\nabla u_n|^2 < \varepsilon . \tag{6.9}$$

Corollary 6.3. *For any bounded cylinder $K = \Omega \times (0, T)$*

$$\nabla u_n \to \nabla u \quad \textit{weakly in } L^2(K) \textit{ as } n \to \infty .$$

The proof of (6.8) is a straightforward generalization of Lemma 1 in [5]. From the same proof, using the continuity at $t = 0$ given in Lemma 6.1, we obtain (6.9). We leave the details to the reader.

In view of Lemma 6.2, our final aim, to prove (6.7) means nothing else than proving that $\nabla u_n \to \nabla u$ strongly in $L^2_{\mathrm{loc}}(\overline{Q})$ as $n \to \infty$. The next lemma is a first result about strong convergence.

Lemma 6.4. *For any* $c > 0$ *let* $P_c \subset Q$ *be defined by*

$$P_c = \{(x, t) \in Q : u(x, t) \geq c\}.$$

Then for any $c > 0$

$$\nabla u_n \to \nabla u \quad \text{in } L^2_{\mathrm{loc}}(P_c) \text{ as } n \to \infty.$$

Lemma 6.4 is an immediate consequence of the monotonicity of u_n (which implies that $u_n \geq c$ in P_c for all n) and the following lemma.

Lemma 6.5.

$$\nabla(u_n^2) \to \nabla(u^2) \quad \text{in } L^2_{\mathrm{loc}}(Q) \text{ as } n \to \infty.$$

Proof. The proof is similar to that of Theorem 1 in [5] provided we show that

$$(6.10) \qquad u_{n_t} \quad \text{is uniformly bounded in } L^2_{\mathrm{loc}}(Q).$$

We multiply (1.1) by

$$\psi(x, t) = \frac{u_{nt}(x, t)}{u_n(x, t)} \phi_1^2(x) \phi_2^2(t)$$

where $\phi_1 \in C_0^\infty(\mathbf{R}^N)$ and $\phi_2 \in C_0^\infty(\mathbf{R})$ are nonnegative functions such that

$$\phi_2(t) = 0 \quad \text{if } t \in (0, \bar{t})$$

for some $\bar{t} > 0$. A straightforward calculation, which uses the estimate $u_{nt} \geq -u_n/t$ given by Proposition 2.2(i), yields that

$$\frac{1}{2} \iint_Q \frac{u_{nt}^2}{u_n} \phi_1^2 \phi_2^2 \\ \leq \frac{\gamma}{\bar{t}} \iint_Q |\nabla u_n|^2 \phi_1^2 \phi_2^2 + \iint_Q |\nabla u_n|^2 \{2u_n |\nabla \phi_1|^2 \phi_2^2 + \phi_1^2 \phi_2 \phi_{2t}\}.$$

Since the right-hand side is bounded, (6.10) follows.

Proof of Proposition 2.1. We have to prove (6.7). Let $\varepsilon > 0$. By (6.9) there exist $t_\varepsilon \in (0, T)$ and $n_\varepsilon > 0$ such that for all $n \geq n_\varepsilon$

$$\int_0^T \int_\Omega ||\nabla u_n|^2 - |\nabla u|^2| < \int_{t_\varepsilon}^T \int_\Omega ||\nabla u_n|^2 - |\nabla u|^2| + \frac{\varepsilon}{3}.$$

Let χ^δ and χ_n^δ be the characteristic functions of the sets $\{(x, t) \in Q: u(x, t) < \delta\}$ and $\{(x, t) \in Q: u_n(x, t) < \delta\}$, respectively, and fix $\alpha \in (0, 1)$. Then, by (6.8),

$$\int_{t_\varepsilon}^T \int_\Omega ||\nabla u_n|^2 - |\nabla u|^2| \tag{6.11}$$
$$\leq 2C_{K,\alpha}\delta^\alpha + \int_{t_\varepsilon}^T \int_\Omega ||\nabla u_n|^2(1-\chi_n^\delta) - |\nabla u|^2(1-\chi^\delta)|.$$

We choose $\delta > 0$ so small that $2C_{K,\alpha}\delta^\alpha < \varepsilon/3$. Finally, because of Lemma 6.4 and the monotonicity of the sequence u_n, there exists an $\overline{n}_\varepsilon \geq n_\varepsilon$ such that the second term at the right-hand side of (6.11) is also smaller than $\varepsilon/3$ for $n \geq \overline{n}_\varepsilon$. Hence

$$\iint_K ||\nabla u_n|^2 - |\nabla u|^2| < \varepsilon \quad \text{if } n \geq \overline{n}_\varepsilon$$

and the proof is complete.

Remark 6.6. The results of this section carry over easily to the case of unbounded initial functions u_0 as long as it is possible to construct the sequence $\{u_n\}$, with $u_n \searrow u$ and u_n uniformly bounded in $L^\infty_{\text{loc}}(\overline{Q})$.

Remark 6.7. In the proof of Proposition 2.1 the monotonicity of the sequence $\{u_n\}$ was used to apply Lemma 6.1 (i.e., with $u_n^{(2)} = u_n$) and to estimate the last term in (6.11). If instead $\{u_n\}$ is a sequence of classical solutions of Problem I which is not necessarily monotone, but such that

$$u_n \to u \quad \text{uniformly on compact subsets of } Q \text{ as } n \to \infty, \tag{6.12}$$

then we can still prove that u is a solution of Problem I and $u \in C(\overline{Q})$, provided that we can apply Lemma 6.1 (i.e., with $u_n^{(1)} = u_n$!). The only thing to check is that the last term of (6.11) vanishes as $n \to \infty$. This follows from the fact that, in view of (6.12) and standard classical theory for uniformly parabolic equations, $u_n \to u$ in $C^{2.1}_{\text{loc}}(P_\delta)$ as $n \to \infty$, where P_δ is defined in Lemma 6.4.

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